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APPLICATION OF THE DISCRETE FOURIER TRANSFORM TO STUDY THE DYNAMICS OF WAVE PACKETS

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A method for solving equations that describe the dynamics of wave packets of the Tollmien–Schlichting waves in the boundary layer is proposed. The method of splitting the initial problem into the linear and nonlinear parts at each time step is used. The linear part is resolved by using an equation for spectral components of the wave packet with a subsequent Fourier transform from the space of wavenumbers to the physical space. A system of ordinary differential equations is solved in the physical space. The Fourier transform is performed by means of the library procedure of the fast Fourier transform. As examples, the problems solved were the linear dynamics of the wave packet concentrated in the vicinity of the instability region (i.e., a set of wave vectors in the space of wavenumbers for which the imaginary part of the eigenfrequency of the Tollmien–Schlichting waves is positive) and the nonlinear dynamics of the wave packet overlapping the instability region.

Key words: *Tollmien–Schlichting waves, Fourier transform, boundary layer.*

Introduction. The boundary layer on a flat plate in an incompressible fluid is considered. Of much interest is the dynamics of disturbances of finite spectral size or wave packets (WP) on the background of the mean flow, because the phenomena proceeding in the laminar part of the boundary layer can have analogs in the turbulent part of the boundary layer. In addition, disturbance-development models can be of interest for transition predictions.

The experiments [1–3] show that the transition from the laminar to the turbulent gas motion in real situations is associated with origination of wave packets (Tollmien–Schlichting waves) of finite spectral size in the laminar part of the boundary layer and their subsequent downstream evolution, which is first linear and then weakly nonlinear. The stage of strong nonlinearity corresponds actually to the transition point. Linear dynamics of wave packets is considered in [4–6]. The early stage of nonlinear dynamics is described in [2, 7–9]. The experimental results of [2, 9] can be used for comparisons in estimating the efficiency of the theoretical models presented in [7, 8].

The disturbance of fluid motion is composed of several parts, in accordance with the types of waves excited in the boundary layer: Tollmien–Schlichting waves and Squire waves (waves of vertical vorticity) with a discrete or continuous spectrum. The influence of the Tollmien–Schlichting waves with a continuous spectrum on disturbance dynamics is considered in [10, 11]. The necessity of taking into account continuous spectrum waves follows, generally speaking, from their weak decay in the long-wave range of the space of wavenumbers. To simplify the problem formulation, continuous spectrum waves are not considered below.

The WP description by the solution of the Navier–Stokes equations is complicated by the small amplitude of the disturbance, as compared to the background flow. Therefore, it is of interest to identify WP dynamics. This becomes possible if we take into account that only one mode of the Tollmien–Schlichting waves is normally excited, which induces the Squire wave (in the case of weak nonlinearity, it can also excite continuous spectrum waves).

In the case of a three-wave resonance, it is difficult to solve nonlinear equations even in a truncated form because of the integrodifferential equation that describes the "0-packet" [7, 8] (new element of dynamics of the three-wave resonance — set of harmonics in the vicinity of the origin of the space of wavenumbers) arising because

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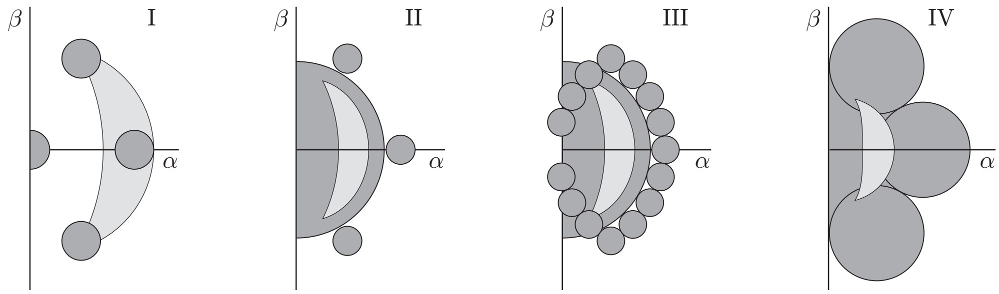

Fig. 1. Possible initial configurations of wave packets.

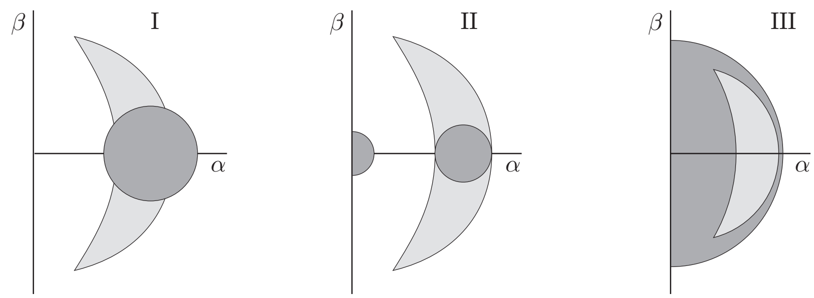

Fig. 2. Simplified initial configurations of wave packets.

of the finite size of wave packets in the weakly nonlinear approximation. Nevertheless, the linear problem of WP dynamics in the space of wavenumbers can be readily solved. In the physical space, the amplitude distribution is obtained by the Fourier transform. Therefore, it seems natural to split the operator of the problem into the linear and nonlinear parts, which are solved in the space of wavenumbers and in the physical space, respectively. The elements of the solution can be coupled by the discrete Fourier transform (fast Fourier transform).

The solution of the spectral (time) problem for the Tollmien–Schlichting waves with a discrete spectrum shows [12] that the set of unstable wavenumbers in the space of wavenumbers α and β is a compact region located near the origin. The continuous spectrum being neglected, the following variants of wave packets with a downstream growing amplitude can be possible (Fig. 1).

Case I. The resonant triplet whose fundamental harmonic is located in the instability region (amplification is described by a numerical coefficient) with the zone containing the origin (singular part of the packet).

Case II. The singularity region contains the instability region (amplification of waves is described by an integral operator); the resonant triplet is located outside the instability region.

Case III. The singularity region contains the instability region; the discrete modes correspond to the multiple three-wave resonance.

Case IV. The singularity region, the resonant triplet, and the instability region overlap (see Fig. 1).

The instability region is marked by the light color in the figure.

Other variants are also possible (e.g., the case of the resonance with a multiple harmonic was considered in [13]). To describe WP dynamics in the cases listed above, one has to know the matrix elements [7, 8] in a wide range of wavenumbers. Hence, these cases are not taken into account for demonstrating the technique operation, and the technique is applied to the following simpler cases (Fig. 2):

1) The wave packets contains the instability region (a linear problem is solved numerically and analytically);

2) The wave packet contains the singularity region and the instability region (a linear problem is solved numerically);

3) The wave packet contains two different components: the singularity region and the discrete harmonic in the amplification region (a nonlinear problem is solved numerically).

1. Equations for Harmonics and Equations for the Wave-Packet Envelope. The equation derived in [7, 8] describes the space–time dynamics of the spectral components $f_{\boldsymbol{k}}$ of a disturbance localized in space:

$$\Big(\frac{\partial}{\partial \bar{t}} + i[\bar{\omega}^R(\boldsymbol{k}) - (\dot{X}_0\boldsymbol{k})]\Big) f_{\boldsymbol{k}} - \varepsilon^2 \frac{\partial}{\partial \alpha}\Big(\frac{\partial \omega^R}{\partial \bar{X}_0} f_{\boldsymbol{k}}\Big) - \varepsilon^2(\bar{\omega}^I(\boldsymbol{k}) + Q(\boldsymbol{k})) f_{\boldsymbol{k}} = \varepsilon \int H_{\boldsymbol{k}\boldsymbol{k}'} f_{\boldsymbol{k}'} f_{\boldsymbol{k}-\boldsymbol{k}'}\, d\boldsymbol{k}' + \ldots \,. \qquad (1)$$

Here, $\bar{\omega}(\boldsymbol{k}) = \bar{\omega}^R(\boldsymbol{k}) + i\bar{\omega}^I(\boldsymbol{k})$ is the dimensionless eigennumber (frequency of eigenoscillations of the Tollmien–Schlichting waves) of the spectral problem for the Orr–Sommerfeld equation, which possesses the instability region in the space of wavenumbers, $\boldsymbol{k}(\alpha, \beta)$ is the wave vector, $k = \sqrt{\alpha^2 + \beta^2}$, $\bar{X}_0(t)$ is the position of the "center of mass" of the localized disturbance (position of the point in the flow, which moves with a certain velocity specified beforehand; the disturbance is observed from a moving coordinate system), the quantities Q and H are specified in [7, 8], and ε is a small parameter.

Hence, the problem of numerical determination of the dynamics of localized disturbances of these types arises. For wave packets of small size, it is possible to obtain a system of partial differential equations in the space of wavenumbers [7, 8], which is, generally speaking, supplemented by an integrodifferential equation for the wave packet located near the origin of the space of wavenumbers ("0-packet"). This circumstance severely complicates the solution of the problem, because the linear integral operator is singular in the physical space. In the space of wavenumbers, however, this problem is not singular. Therefore, an idea arises to split the operator in the course of the numerical solution into the linear and nonlinear parts.

Since the amplification region is located near the origin ($k \approx 0.1$–0.3), it is of interest to approximate the eigenfrequency in this region. Approximation of the dimensionless frequency in the vicinity of zero has the following form [7, 8, 12]:

$$\bar{\omega}(\boldsymbol{k}) = \bar{\omega}^R(\boldsymbol{k}) + i\varepsilon^2\bar{\omega}^I(\boldsymbol{k}) = \alpha a(\bar{X}_0) + b(\bar{X}_0)\alpha k + i\varepsilon^2 d(\bar{X}_0).$$

The parameters of this expression are defined in [12].

Figure 1 shows some possible initial configurations of the spectral components on the background of the amplification region. The spectral size of an individual wave packet composing the initial configuration can be small (case I). The dynamics of such wave packets in the physical space is determined by their small vicinity in the space of wavenumbers. If the wave packet contains the amplification region (case IV), the disturbance amplification (decay) in the physical space is determined by an integral operator covering the entire region where the localized disturbance is concentrated. Intermediate cases (cases II and III) are also possible.

2. Narrow Wave Packets. Equations of dynamics of the wave-packet envelope in the laminar part of the boundary layer on a flat plate were derived in [7, 8]. They include an integrodifferential equation, which describes the dynamics of wave harmonics with the wave vector in the vicinity of the origin of the space of wavenumbers, which, in the general case, are subjected to self-action and interaction with a set of wave packets that are in a three-wave resonance with each other. In turn, the dynamics of these wave packets described by differential equations of the type of the nonlinear Schrödinger equation is affected by wave harmonics from the vicinity of the origin of the space of wave vectors. These equations are

$$\frac{\partial \varphi^{(0)}}{\partial t}(t, \boldsymbol{r}_1) - \varepsilon(\dot{\bar{X}}_0 - a(\bar{X}_0))\, \frac{\partial \varphi^{(0)}}{\partial x_1}(t, \boldsymbol{r}_1) = \varepsilon^2 I^{(0)}; \qquad (2)$$

$$\frac{\partial \tilde{\varphi}_j^{(n)}}{\partial t} + i(q(\hat{P}_j^{(n)}) - q(n\boldsymbol{k}_j))\tilde{\varphi}_j^{(n)} + i\varepsilon x_1 \frac{\partial \omega^R}{\partial \bar{X}_0}\, (\hat{P}_j^{(n)})\tilde{\varphi}_j^{(n)} = \varepsilon^2 I_j^{(n)}, \qquad (3)$$

where

$$I^{(0)} = -b(\bar{X}_0) \int G(\boldsymbol{r}_1 - \boldsymbol{s})\, \frac{\partial}{\partial \xi}\Big(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\Big)\varphi^{(0)}(t, \boldsymbol{s})\, d\boldsymbol{s} - ix_1\, \frac{\partial a(\bar{X}_0)}{\partial \bar{X}_0}\, \frac{\partial \varphi^{(0)}}{\partial x_1}(t, \boldsymbol{r}_1)$$

$$+ (d(\bar{X}_0) + Q(0))\varphi^{(0)}(t, \boldsymbol{r}_1) + \sum_{j=1}^{3} \sum_{\substack{n=-1\\ n\neq 0}}^{1} H_{0,n\boldsymbol{k}_j} \tilde{\varphi}_j^{(n)}(t, \boldsymbol{r}_1)\tilde{\varphi}_j^{(-n)}(t, \boldsymbol{r}_1) + H_{0,0}(\varphi^{(0)}(t, \boldsymbol{r}_1))^2;$$

$$G(\boldsymbol{r}) = 1/(\pi r);$$

$$I_j^{(n)} = (\bar\omega^I(\hat P_j^{(n)}) + Q(\hat P_j^{(n)}))\tilde\varphi_j^{(n)}(t,\boldsymbol{r}_1) + ix_1^2 P^{(2)}(\hat P_j^{(n)})(n\alpha_j)\tilde\varphi_j^{(n)}(t,\boldsymbol{r}_1)$$

$$+ \sum_{l=-1}^{1} H(n\boldsymbol{k}_j, l\boldsymbol{k}_j)\tilde\varphi_j^{(l)}(t,\boldsymbol{r}')\tilde\varphi_j^{(n-l)}(t,\boldsymbol{r}'')e_j^{(l)}e_j^{(n-l)}e_j^{(-n)} + V_j^{(n)};$$

$$\hat P_j^{(n)} = n\boldsymbol{k}_j + (\varepsilon/i)\nabla; \qquad q(\boldsymbol{k}) = \bar\omega^R(\boldsymbol{k}) - \dot{\bar X}_0\alpha; \qquad \varphi_j^{(n)} = e_j^{(n)}\tilde\varphi_j^{(n)};$$

$$\boldsymbol{s} = (\xi,\eta); \qquad e_j^{(n)} = \exp\Big(-i\int_0^t q(n\boldsymbol{k}_j)\,dt\Big); \qquad l,n = -1,0,1; \quad j = 1,2,3.$$

In the last formula, $V_j^{(n)}$ are determined by the expressions

$$V_1^{(n)} = \{H(n\boldsymbol{k}_1, n\boldsymbol{k}_2) + H(n\boldsymbol{k}_1, n\boldsymbol{k}_3)\}\tilde\varphi_2^{(n)}(\boldsymbol{r})\tilde\varphi_3^{(n)}(\boldsymbol{r});$$

$$V_2^{(n)} = \{H(n\boldsymbol{k}_2, n\boldsymbol{k}_1) + H(n\boldsymbol{k}_2, n\boldsymbol{k}_3)\}\tilde\varphi_1^{(n)}(\boldsymbol{r})\tilde\varphi_3^{(-n)}(\boldsymbol{r});$$

$$V_3^{(n)} = \{H(n\boldsymbol{k}_3, n\boldsymbol{k}_1) + H(n\boldsymbol{k}_3, n\boldsymbol{k}_2)\}\tilde\varphi_1^{(n)}(\boldsymbol{r})\tilde\varphi_2^{(-n)}(\boldsymbol{r}).$$

Here $\tilde\varphi_j^{(n)}(\boldsymbol{r})$ ($n = -1, 1$ and $j = 1,2,3$), $\boldsymbol{r} = (x,z)$ is the amplitude of the envelope of the resonant triplet component, $\tilde\varphi_j^{(-n)}$ are complex conjugate with $\tilde\varphi_j^{(n)}$, $\varphi^{(0)}$ is the amplitude of the "0-packet," and $\boldsymbol{k}_1, \boldsymbol{k}_2$, and $\boldsymbol{k}_3$ are the wave vectors of the resonant triplet [$\boldsymbol{k}_1 = \boldsymbol{k}_2 + \boldsymbol{k}_3$, $\bar\omega^R(\boldsymbol{k}_1) = \bar\omega^R(\boldsymbol{k}_2) + \bar\omega^R(\boldsymbol{k}_3)$]; the remaining designations are the same as in [8]. The terms corresponding to multiple harmonics are ignored in Eqs. (2) and (3).

The left side of the operator of the equations for $n \neq 0$ has the following form with accuracy to $O(\varepsilon^2)$:

$$\frac{\partial\tilde\varphi_j^{(n)}(\boldsymbol{r},t)}{\partial t} + \varepsilon\Big[\frac{\partial\bar\omega^R}{\partial\boldsymbol{k}}\Big|_{\boldsymbol{k}=n\boldsymbol{k}_j}\cdot\nabla - \dot X_0\frac{\partial}{\partial x}\Big]\tilde\varphi_j^{(n)}(\boldsymbol{r},t) - i\,\frac{\varepsilon^2}{2}\Big(\nabla\cdot\frac{\partial}{\partial\boldsymbol{k}}\Big)^2\bar\omega^R\Big|_{\boldsymbol{k}=n\boldsymbol{k}_j}\tilde\varphi_j^{(n)}(\boldsymbol{r},t),$$

$$\nabla = \Big(\frac{\partial}{\partial x},\frac{\partial}{\partial z}\Big), \qquad \frac{\partial}{\partial\boldsymbol{k}} = \Big(\frac{\partial}{\partial\alpha},\frac{\partial}{\partial\beta}\Big).$$

The value of $H_{0,0}$ was determined [12] as an intermediate limit $H_{\boldsymbol{k},\boldsymbol{k}-\boldsymbol{k}_1}$, where $\boldsymbol{k}$ is directed along the longitudinal axis of the space of wavenumbers; $\boldsymbol{k}_1$ and $\boldsymbol{k}-\boldsymbol{k}_1$ are its subharmonics with a small but not vanishing $\boldsymbol{k}$. It turned out that $\mathrm{Re}\,[H_{\boldsymbol{k},\boldsymbol{k}-\boldsymbol{k}_1}]$ and $\mathrm{Im}\,[H_{\boldsymbol{k},\boldsymbol{k}-\boldsymbol{k}_1}]$ behave as power functions of $\alpha^{0.72}$ and $\alpha^{0.45}$, respectively, as $\alpha \to 0$. Renormalizing the initial equations ($f_{\boldsymbol{k}} = k^\mu \tilde f_{\boldsymbol{k}}$ and $\mu \cong -0.72$), we can make the new value of $H_{0,0}$ finite and real, which regularizes the equations for the wave-packet envelope derived in [7, 8].

3. Some Particular Cases. In what follows, we consider cases where the longitudinal inhomogeneity is neglected. In case 1, the instability region can be approximated by a polynomial. In this case, the dynamics of the initially Gaussian wave packet is described by the quadrature

$$\Psi(x,z,t) = \int_{-\infty}^{\infty} d\beta \int_{-\infty}^{\infty} d\alpha \exp\Big(-it\Phi_1 + t\Phi_2 - \Phi_3 + i((\alpha-0.3)x + \beta z)\Big)$$

$$\cong 0.293\exp\Big[-0.25i(12 + (0.3+0.6ib)t + ix)^2/((b-i)t - 20i) - (0.0125 + 0.09ib)t$$

$$-0.3ix + 0.5iz^2/((b-0.4i)t - 40i)\Big]\Big/\sqrt{(20+i+ibt)(6.366 + 0.637t + 0.159ibt)},$$

$$\Phi_1 = (2b(\alpha-0.3)^2 + b\beta^2)/2, \quad \Phi_2 = 0.01 - (\alpha-0.15)^2 - \beta^2/5, \quad \Phi_3 = ((\alpha-0.3)^2 + \beta^2)/0.05.$$

The calculation results are plotted in Fig. 3.

The same problem can be solved numerically in the linear approximation with the help of the discrete Fourier transform. A comparison with the experimental results of [4, 9] reveals qualitative agreement with analytical and numerical results of the present work.

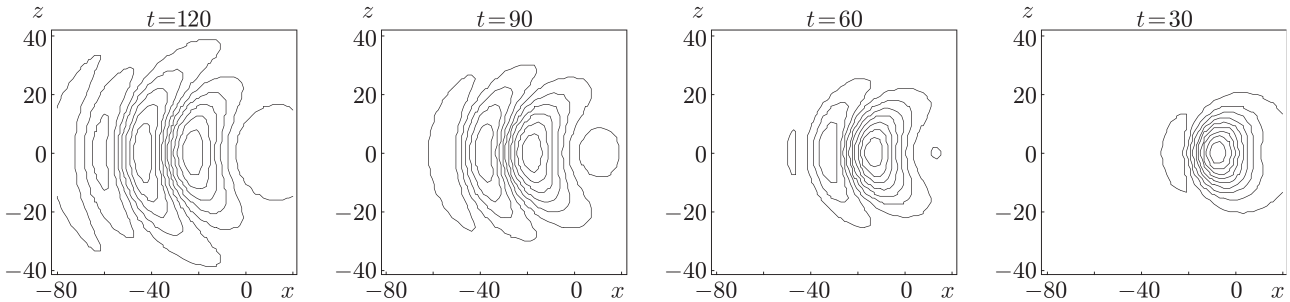

Fig. 3. Linear dynamics of the Gaussian wave packet.

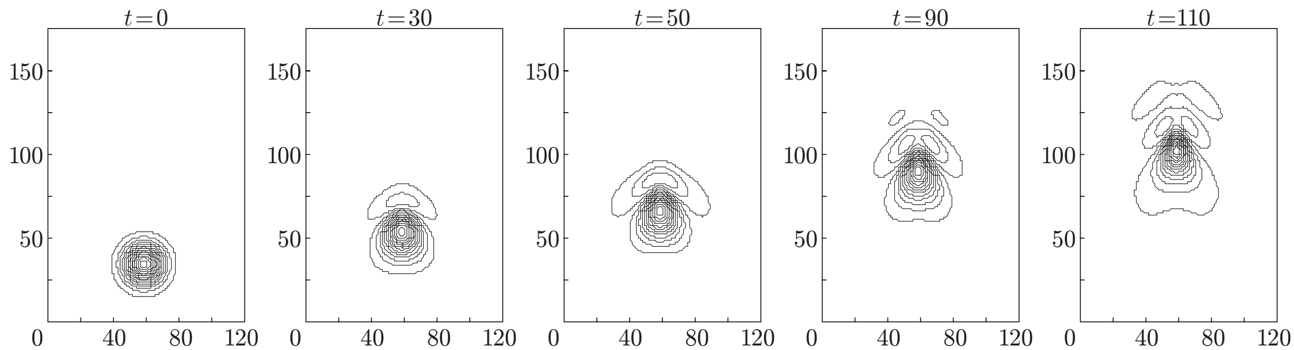

Fig. 4. Dynamics of the wave packet with self-action (the horizontal and vertical directions correspond to the z and x axes).

The numerical solution allows us to consider the case where the WP carrier contains the origin of the space of wavenumbers and the instability region (case 3). Self-action is taken into account. The splitting scheme has the following form:

$$\frac{\partial \varphi^{(0)}}{\partial t} = H_{0,0}(\varphi^{(0)})^2, \qquad \frac{\partial f_{\boldsymbol{k}}}{\partial t} = -i\Omega f_{\boldsymbol{k}}.$$

Here $i\Omega = (i[\bar{\omega}^R(\boldsymbol{k}) - (\dot{X}_0\boldsymbol{k})]) - \varepsilon^2(\bar{\omega}^I(\boldsymbol{k}) + Q(\boldsymbol{k}))$. The calculation results show that the initially Gaussian WP transforms to a wave packet localized in the instability region. After that, the stage of WP dispersion in the physical space begins. Possibly, this result can explain the phenomenon described in [14, 15] and in monograph [2], where transformation of the initially long-wave WP (streaky structures) to a short-wave WP was observed.

The solution of this problem is shown in Fig. 4. It should be noted that allowance for subharmonic components in the initial wave packet in the numerical example given can only increase the agreement with experimental results.

The results obtained allow us to hope that the method proposed can describe the wave packet with a more complicated initial structure.

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REFERENCES

1. H. W. Emmons, "The laminar–turbulent transition in a boundary layer. Part I," *J. Aeronaut. Sci.*, **18**, No. 7, 490–498 (1951).
2. A. V. Boiko, G. R. Grek, A. V. Dovgal', and V. V. Kozlov, *Origination of Turbulence in Near-Wall Flows* [in Russian], Nauka, Novosibirsk (1999).

3. B. Yu. Zanin, “Transition to turbulence on an airfoil in flight and in a wind tunnel with identical Reynolds numbers,” *Izv. Sib. Otd. Akad. Nauk SSSR, Ser. Tekh. Nauk*, **18**, No. 5, 51–53 (1988).
4. M. Gaster and I. Grant, “An experimental investigation of the formation and development of wave packet in a laminar boundary layer,” *Proc. Roy. Soc. London*, **A347**, 253–269 (1975).
5. E. D. Terent'ev, “Formation of a wave packet in the boundary layer on a flat plate,” *Prikl. Mat. Mekh.*, **51**, No. 5, 814–819 (1987).
6. O. S. Ryzhov and I. V. Savenkov, “Asymptotic theory of the wave packet in the boundary layer on a flat plate,” *Prikl. Mat. Mekh.*, **51**, No. 5, 820–828 (1987).
7. V. A. Zharov, “Asymptotic description of weakly nonlinear wave packets in a medium with weak dispersion typical of laminar boundary layers on a flat place in an incompressible fluid,” *Tr. TsAGI*, No. 2523, 3–16 (1993).
8. V. A. Zharov, “Variant of the description of weakly nonlinear dynamics of the wave packet in the boundary layer on a flat plate in an incompressible fluid,” *ibid.*, pp. 17–28.
9. M. A. F. Medeiros and M. Gaster, “The production of subharmonic waves in the nonlinear evolution of wave packets in boundary layers,” *J. Fluid Mech.*, **399**, 301–318 (1999).
10. V. N. Zhigulev, N. V. Sidorenko, and A. M. Tumin, “Generation of instability waves in the boundary layer by external turbulence,” *Prikl. Mekh. Tekh. Fiz.*, **21**, No. 6, 43–49 (1980).
11. V. P. Reutov and G. V. Rybushkina, “Excitation of continuous spectrum wave packets in the boundary layer by external turbulence,” Preprint, Inst. Appl. Phys., Russ. Acad. of Sci., Nizhii Novgorod (1992).
12. I. G. Dodonov, V. A. Zharov, and Yu. I. Khlopkov, “Localized coherent structures in the boundary layer,” *J. Appl. Mech. Tech. Phys.*, **41**, No. 6, 1012–1019 (2000).
13. V. P. Reutov, “Nonlinear critical layer and formation of linear vortices with reaction of waves in shear flows,” *J. Appl. Mech. Tech. Phys.*, **28**, No. 5, 742–749 (1987).
14. G. D. Grek, V. V. Kozlov, and M. P. Ramazanov, “Three types of disturbances from the point source in the boundary layer,” in: *Laminar–Turbulent Transition*, Springer-Verlag, Berlin (1985), pp. 267–272.
15. G. D. Grek, J. Dey, V. V. Kozlov, et al., “Experimental analysis of the process of the formation of turbulence in the boundary layer at higher degree of turbulence of windstream,” TR 91-FM-2, Indian Inst. of Science, Banglador (1991).